

The University of Chicago

INVARIANT MULTIPLE
INTEGRALS IN THE CALCULUS OF
VARIATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1939

By

AUBREY WILFRED LANDERS, JR.

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TABLE OF CONTENTS

Section	Page
Introduction	1
1. Formulation of the problem	3
2. Invariant integrals	6
3. Investigation of fields by the use of specialized coordinates	11
4. Results expressed in terms of the original coordinates	18
5. Certain systems of partial differential equations .	22
Bibliography	29

INVARIANT MULTIPLE INTEGRALS IN THE CALCULUS OF VARIATIONS

Introduction. The calculus of variations problem with which we are concerned is that of minimizing a multiple integral of the form

$$\int_A f(t_1, \dots, t_m; x^1, \dots, x^n; x_1^1, \dots, x_m^n) dt_1 \dots dt_m$$

in a class of admissible n -dimensional spaces $x = x(t)$, for each of which the integral has a well defined value, and all of which pass through a fixed $(n-1)$ -dimensional bounding space L in tx -space. In this integral x^i denotes the unknown function $x^i(t)$ and x_a^i stands for the partial derivative of x^i with respect to t_a , while A is the region of integration in t -space.

The theory of fields and their associated invariant integrals for multiple integral problems have been studied from different points of view by Bliss (I)¹, Caratheodory (II) and Weyl (III). In the paper by Bliss referred to here, he restricts himself to double integral problems but, more recently, he has given a general definition of a field for the problem here studied. Dr. M. F. Smiley, while studying under Professor Bliss, obtained a set of conditions which every integrand of an invariant integral must satisfy. In this paper we show, by a method analogous to that used by Bliss (I), that these same conditions on the integrand are

¹Roman numerals in parenthesis refer to the bibliography at the end of the paper.

sufficient to make an integral invariant. We then make a study of fields having associated with them invariant integrals of this general type. Solutions of certain systems of partial differential equations which arise in this study are found by a method suggested by Hestenes.

In the first section of this paper we formulate the problem in detail, express the Euler and Weierstrass necessary conditions, define a field and show what would be needed to make a sufficiency proof for minimizing the above integral. In the second section we give necessary and sufficient conditions for the value of an integral of the above type to depend only on the bounding space of the class of admissible spaces under consideration.

It is well known that when $n = m = 1$ every family of extremals which simply covers a region F in tx -space defines a unique field over F . When $n > m = 1$ a family of extremals may simply cover a region F without defining a field. However, if such a family defines a field over F , it is unique. In Sections 3 and 4 below, it is shown that when $m > 1$ every family of extremals, with suitable analytic properties, which simply covers a region F defines a field over F . This field is unique only in case $n = 1$. The form of every invariant integral determined by a family of extremals of this type is also given. When $m > 1$, $n > 1$ the most general invariant integral associated with a field is expressed in terms of solutions of systems of linear partial differential equations of the first order. Solutions of these equations are given in the final section. The above results are first obtained by specializing the coordinates in the manner described in Section 3. In Section 4, they are interpreted in terms of the original coordinates. As a by-product, we obtain the usual necessary and sufficient conditions for the formation of a field when $n > m = 1$.

1. Formulation of the problem. Let us consider the m -tuple integral

$$(1:1) \quad I = \int_T f(t, x, x_t) dt$$

where

$$t \equiv (t_1, \dots, t_m), \quad x \equiv (x^1, \dots, x^n), \quad x_t \equiv (x_1^1, \dots, x_m^n),$$

$$dt \equiv dt_1 \dots dt_m, \quad x_a^1 \equiv \partial x^1 / \partial t_a \quad (a = 1, \dots, m; 1 = 1, \dots, n)$$

and T is an m -dimensional region in t -space. The integrand function f is assumed to have continuous partial derivatives, of as many orders as is needed for the arguments of the following pages, in a region R of an $(m+n+mn)$ -dimensional space of elements (t, x, x_t) .

An m -space defined by functions of the form

$$(1:2) \quad x(t) \quad (t \text{ in } T)$$

will be said to be admissible if the functions $x(t)$ have continuous partial derivatives of the second order in a bounded closed connected region T of t -space and if further all the elements (t, x, x_t) belonging to the space are interior to R .

Let A denote the $(m-1)$ -space in t -space which bounds T and let L represent a fixed $(m-1)$ -space in tx -space whose projection on t -space is A . The points of A and L are to be in one-to-one correspondence. The calculus of variations problem is then to find in the class of admissible m -spaces (1:2) which pass through a given $(m-1)$ -space L , one which minimizes the integral (1:1). It is understood that in the integrand of the integral I the variable x^1 is replaced by $x^1(t)$ from (1:2) and x_a^1 is replaced by $dx^1(t)/dt_a$. Here and in the following pages the index a with or without a subscript has the range of integers

from 1 to m , and the index i similarly the range from 1 to n .

The Euler necessary condition for the above problem may be stated as follows (VII, p. 17):

THEOREM 1:1. On a minimizing m -space $x = x(t)$, (t in T), we must have

$$(1:3) \quad \frac{df_{x_a^i}}{dt} = f_{x_a^i}.$$

As in tensor analysis a repeated index a or i , with or without a subscript, indicates summation over the specified range. An admissible m -space which satisfies the Euler equations (1:3) is called an extremal.

For the above problem the E-function of Weierstrass is expressed by the equation

$$E(t, x, x_t, X_t) = f(t, x, X_t) - f(t, x, x_t) - (X_a^1 - x_a^1) f_{x_a^1}(t, x, x_t).$$

The Weierstrass necessary condition may be stated as follows:¹

THEOREM 1:2. At every element (t, x, x_t) of a minimizing m -space $x = x(t)$, (t in T), the condition

$$E(t, x, x_t, X_t) \geq 0$$

is satisfied for every X_t such that the matrix $\|X_a^1 - x_a^1\|$ is of rank 1.

Bliss has given the following definition of a field. A field for the integral I is a region F of tx -space with which there is associated a set of slope functions $p_a^1(t, x)$ having continuous partial derivatives of the second order in F and an m -tuple integral

$$(1:4) \quad I^* = \int_T D(t, x, x_t) dt$$

¹McShane (VI, p. 584); for $n = 2$, Coral (IV, p. 589).

such that

$$(1:5) \quad D(t, x, p) = f(t, x, p), \quad D_{x_a^1}(t, x, p) = f_{x_a^1}(t, x, p)$$

and further such that the value of I^* is the same for all admis-
sible m -spaces $x = x(t)$ in F with a common boundary L .

The differential equations of the field are

$$(1:6) \quad x_a^1 = p_a^1(t, x) \quad (t \text{ in } T).$$

Every solution of these equations is an extremal, for, if we differentiate the first of equations (1:5) with respect to x^1 and use the other equations of (1:5) we find

$$D_{x^1}(t, x, p) = f_{x^1}(t, x, p)$$

and hence

$$(d/dt_a)f_{x_a^1}(t, x, p) - f_{x_a^1}(t, x, p) = (d/dt_a)D_{x_a^1}(t, x, p) - D_{x_a^1}(t, x, p) = 0.$$

A solution of equations (1:6) is called an extremal of the field. We see by means of the first equation of (1:5) that on an extremal space of the field the values of I and I^* are equal.

If we denote by $I(E)$ and $I(S)$, respectively, the values of I computed for an extremal space E of the field with boundary L and an admissible m -space S of the form (1:2) with the same boundary, then

$$I(S) - I(E) = I(S) - I^*(E) = I(S) - I^*(S) = \int_T (f - D) dt$$

where (t, x, x_t) in the integrand function are those belonging to S . It is evident that if there exists an invariant integral of the form (1:4) such that $f - D \geq 0$, then E minimizes the integral I .

2. Invariant integrals. Let R_0 represent a region of admissible elements (t, x, x_t) characterized by conditions of the form

$$t \text{ in } T; \quad X^{11}(t) < x^1 < X^{s1}(t); \quad x_t \text{ arbitrary,}$$

where $X^1(t)$ and $X^s(t)$ are admissible m -spaces. We shall give a proof of the following theorem:

THEOREM 2:1. An integral I of the form (1:1) has the same value for all m -spaces lying in a region R_0 of the type described above and having the same fixed $(m-1)$ -space L as boundary if and only if the integrand $f(t, x, x_t)$ of I is expressible in R_0 in the form

$$(2:1) \quad f(t, x, x_t) = B(t, x) + \sum_{r=1}^s \frac{1}{r!} x_{a_1}^{i_1} \dots x_{a_r}^{i_r} B_{i_1 \dots i_r}^{a_1 \dots a_r}(t, x)$$

where s is the smaller of m and n , and where

$$(2:2) \quad (\partial / \partial t_a) B_{i_1 \dots i_r}^{a a_1 \dots a_r} = B_{i_1 \dots i_r}^{a_1 \dots a_r} \quad (r=0, 1, \dots, s)$$

in which

$$(2:3) \quad \begin{aligned} B_{i_1 \dots i_{r1}}^{a_1 \dots a_r} &= (\partial / \partial x^1) B_{i_1 \dots i_r}^{a_1 \dots a_r} - (\partial / \partial x^{11}) B_{i_1 \dots i_{r-1} 1}^{a_1 a_2 \dots a_r} \\ &- \dots - (\partial / \partial x^{1r}) B_{i_1 \dots i_{r-1} 1}^{a_1 \dots a_{r-1} a_r} \end{aligned}$$

and the interchange of two a 's or two i 's simply changes the sign of the B 's. It is understood here that $B_{i_1 \dots i_r}^{a_1 \dots a_r}$ reduces to B for $r=0$.

A proof that the conditions of this theorem are necessary was made by Dr. M. F. Smiley while a student in a class conducted by Professor Bliss. For the sake of completeness this proof is given below. It will be proved, by a method analogous to that used by Bliss for the double integral problem (I, pp. 6-8), that these same conditions are sufficient.

In order to prove the necessity of the conditions stated in the theorem, we first note that for every prescribed set $(t, x, x_t, x_{tt})_0$ with $(t, x, x_t)_0$ in R_0 and t_0 interior to T there is an admissible m -space $x = x(t)$ passing through the prescribed set. Let L be an $(m-1)$ -space, of the type described in the preceding section, on this admissible m -space and having the point $(t, x)_0$ in the interior of the region bounded on the space $x(t)$ by the set L . Since the integral I has the same value on all m -spaces through the $(m-1)$ -space L , it follows that all such m -spaces are minimizing spaces. In particular the m -space $x = x(t)$ constructed above has this property and the Euler equations (1:3) must be satisfied on it. These equations may be written in the form

$$(2:4) \quad f_{x_a^i t_a} + x_a^j f_{x_a^i x_a^j} + x_{ba}^j f_{x_a^i x_b^j} = f_{x^1}.$$

Here and in the following pages it will be convenient, at times, to use some or all of the letters b and c in place of a 's with subscripts and j, k and h in place of i 's with subscripts. Since the set $(t, x, x_t, x_{tt})_0$ can be selected arbitrarily it follows that equations (2:4) hold at every set (t, x, x_t, x_{tt}) with (t, x, x_t) admissible and (t, x) in R_0 . Hence

$$(2:5) \quad f_{x_a^i x_b^j} + f_{x_b^j x_a^i} = 0$$

for (t, x, x_t) admissible and (t, x) in R_0 . It follows from equations (2:5) that $f_{x_{a_1}^{i_1} \dots x_{a_r}^{i_r}}$ vanishes whenever indices a or i are repeated, and consequently that f is of degree at most s in x_a^i , where s is the smaller of m and n . Thus for (t, x, x_t) admissible and (t, x) in R_0 , the integrand function f has the form given by equation (2:1) where the B 's in this equation are functions of t and x with continuous partial derivatives of as many orders as are possessed by

f in R_0 . It is evident from the form of equation (2:1) that for (t, x) in R_0 ,

$$(2:6) \quad f_{x_{a_1}^{i_1} \dots x_{a_r}^{i_r}}(t, x, 0) = B_{i_1 \dots i_r}^{a_1 \dots a_r}(t, x).$$

It follows from equations (2:5) and (2:6) that the interchange of two a 's or two i 's simply changes the sign of $B_{i_1 \dots i_r}^{a_1 \dots a_r}$ with $r > 1$ and hence further that these same B 's vanish identically for two a 's or two i 's equal. In particular, all of these B 's in which $r > s$ vanish identically. Examination of the identities (2:3) shows that these same properties apply to $B_{i_1 \dots i_r}^{a_1 \dots a_r}$. It follows that whenever $m \geq n$ the equations (2:2) with $r = s$ are always satisfied, since at least two of the i 's in the set $(i_1, i_2, \dots, i_s)$ are equal.

From equation (2:1) we find that

$$f_{x_a^i} = B_1^a + \sum_{r=2}^s \frac{1}{(r-1)!} x_{a_2}^{i_2} \dots x_{a_r}^{i_r} B_{i_1 i_2 \dots i_r}^{a a_2 \dots a_r}.$$

By means of these equations and the relation (2:1), equations (2:4) become

$$\begin{aligned} & (\partial / \partial t_a) B_1^a + \sum_{r=2}^s \frac{1}{(r-1)!} x_{a_2}^{i_2} \dots x_{a_r}^{i_r} (\partial / \partial t_a) B_{i_1 i_2 \dots i_r}^{a a_2 \dots a_r} \\ & + x_a^{i_1} (\partial / \partial x^{i_1}) B_1^a + x_a^{i_1} \sum_{r=2}^s \frac{1}{(r-1)!} x_{a_2}^{i_2} \dots x_{a_r}^{i_r} (\partial / \partial x^{i_1}) B_{i_1 i_2 \dots i_r}^{a a_2 \dots a_r} \\ & - (\partial / \partial x^i) B = \sum_{r=1}^s \frac{1}{r!} x_{a_1}^{i_1} \dots x_{a_r}^{i_r} (\partial / \partial x^i) B_{i_1 \dots i_r}^{a_1 \dots a_r} = 0. \end{aligned}$$

In the first summation, we replace r by $r+1$ and then rename the indices a_{r+1} and i_{r+1} , respectively, a_1 and i_1 . In the second summation, we may bring in the preceding term by including $r = 1$, rename a as a_1 and change the form as indicated below so that the factor $1/r!$ will appear. Using the functions defined by relations

(2:3) we now have

$$\begin{aligned} (\partial/\partial t_a)B_1^a - (\partial/\partial x^i)B + \sum_{r=1}^s \frac{1}{r!} x_{a_1}^{i_1} \dots x_{a_r}^{i_r} \left[(\partial/\partial t_a)B_{1i_1 \dots i_r}^{a a_1 \dots a_r} \right. \\ \left. - B_{1i_1 \dots i_r 1}^{a_1 \dots a_r} \right] = 0. \end{aligned}$$

Since these equations must hold for (t, x) in R_0 and x_t arbitrary, the equations (2:2) follow at once, thus completing the proof of the necessity of the conditions stated in the theorem.

When the integrand function f has the form described in the theorem, we let S_1 and S_2 be two admissible m -spaces in R_0 with the same bounding $(m-1)$ -space L and defined by the equations

$$x^i = x^{1i}(t), \quad x^i = x^{2i}(t) \quad (t \text{ in } T).$$

The function $z(t)$ will be defined by the equations

$$(2:7) \quad z^i(t) = x^{2i}(t) - x^{1i}(t).$$

The family of admissible m -spaces

$$x^i = x^{1i}(t) + ez^i(t)$$

then contains S_1 for $e = 0$ and S_2 for $e = 1$. The derivative of the m -tuple integral

$$I(e) = \int_T f(t, x + ez, x_t + ez_t) dt$$

with respect to e has the value

$$I'(e) = \int_T (z^1 f_{x^1} + z^2 f_{x^2}) dt.$$

Since the Euler equations (1:3) for the integrand function f are by hypothesis identically satisfied it follows that this derivative has also the value

$$I'(e) = \int_T [d(z^1 f_{x_a^1})/dt_a] dt.$$

By means of an extended Green's Theorem¹, this last integral is equal, except perhaps for sign, to the $(m-1)$ -tuple integral

$$(2:8) \quad \int_A z^1 f_{x_a^1} d_a t \quad (d_a t \equiv dt_1 \cdots dt_{a-1} dt_{a+1} \cdots dt_m).$$

Since the two admissible spaces S_1 and S_2 have the same boundary L , equation (2:7) shows that z^1 vanishes identically over A and hence the value of the integral (2:8) is zero. Thus $I(0)$ and $I(1)$ are equal, and since S_1 and S_2 can be selected arbitrarily from the class of admissible m -spaces in R_0 having the same fixed $(m-1)$ -space L as boundary, the theorem is proved.

In order to show more clearly the nature of the conditions (2:2), we shall write them in more detail for a few special cases.

When $s = 2$, the conditions (2:2) become

$$(2:9) \quad \partial B_1^a / \partial t_a = \partial B / \partial x^1$$

$$(2:10) \quad \partial B_{1j}^{ab} / \partial t_a = \partial B_j^b / \partial x^1 - \partial B_1^b / \partial x^j$$

$$(2:11) \quad \partial B_{jk}^{ab} / \partial x^1 + \partial B_{k1}^{ab} / \partial x^j + \partial B_{1j}^{ab} / \partial x^k = 0$$

If $m = 2$ and $n > 2$, these conditions may be written

$$\partial B_1^1 / \partial t_1 + \partial B_1^2 / \partial t_2 = \partial B / \partial x^1$$

$$\partial B_{1j}^{12} / \partial t_1 = \partial B_j^2 / \partial x^1 - \partial B_1^2 / \partial x^j, \quad \partial B_{1j}^{12} / \partial t_2 = \partial B_1^1 / \partial x^j - \partial B_j^1 / \partial x^1$$

$$\partial B_{jk}^{12} / \partial x^1 + \partial B_{k1}^{12} / \partial x^j + \partial B_{1j}^{12} / \partial x^k = 0$$

¹ P. Franklin, Multiple integrals in n -space, Annals of Mathematics, Second Series, vol. 24 (1922-23), p. 217.

If $n = 2$ and $m \geq 2$, the remarks below equations (2:6) show that equations (2:11) are always satisfied. In this case, equations (2:9) and (2:10) may be written

$$\partial B_1^a / \partial t_a = \partial B / \partial x^1, \quad \partial B_2^a / \partial t_a = \partial B / \partial x^2$$

$$\partial B_{12}^{ab} / \partial t_a = \partial B_2^b / \partial x^1 - \partial B_1^b / \partial x^2.$$

In particular, in this last case when $m = 2$, we have

$$\partial B_1^1 / \partial t_1 + \partial B_1^2 / \partial t_2 = \partial B / \partial x^1, \quad \partial B_2^1 / \partial t_1 + \partial B_2^2 / \partial t_2 = \partial B / \partial x^2$$

$$\partial B_{12}^{12} / \partial t_2 = \partial B_1^1 / \partial x^2 - \partial B_2^1 / \partial x^1, \quad \partial B_{12}^{12} / \partial t_1 = \partial B_2^2 / \partial x^1 - \partial B_1^2 / \partial x^2.$$

When $s = 3$, the conditions (2:2) become

$$\partial B_1^a / \partial t_a = \partial B / \partial x^1$$

$$\partial B_{1j}^{ab} / \partial t_a = \partial B_j^b / \partial x^1 - \partial B_1^b / \partial x^j$$

$$\partial B_{1jk}^{abc} / \partial t_a = \partial B_{jk}^{bc} / \partial x^1 + \partial B_{k1}^{bc} / \partial x^j + \partial B_{1j}^{bc} / \partial x^k$$

$$\partial B_{jkh}^{abc} / \partial x^1 - \partial B_{khi}^{abc} / \partial x^j + \partial B_{hij}^{abc} / \partial x^k - \partial B_{1jk}^{abc} / \partial x^h = 0.$$

3. Investigation of fields by the use of specialized coordinates. Let the equations

$$(3:1) \quad x^1 = X^1(t, c) \quad [(t, c) \text{ in } G]$$

be a family of extremals depending upon n parameters $c = (c^1, \dots, c^n)$ such that the functions X^1, X_a^1 belonging to the family have continuous partial derivatives of the third order at points (t, c) in a region G whose projection in $(t_1, \dots, t_m)$ -space is convex. The extremals (3:1) are said to simply cover a region F of points (t, x) if to each point of the region there corresponds one and

only one solution (t, c) of equations (3:1) in the region G , and if furthermore the determinant $|X_{oj}^1|$ is different from zero at each such solution. These solutions then define a set of single-valued functions $c^1(t, x)$ which have continuous partial derivatives of the third order in F . The functions

$$p_a^1(t, x) = X_a^1[t, c(t, x)]$$

are called the slope functions of the family in F .

By placing

$$(3:2) \quad y^1 = c^1$$

in equations (3:1) we have a transformation

$$(3:3) \quad x^1 = X^1(t, y)$$

by which the coordinates (t, x) are changed to (t, y) . Differentiation with respect to t_a of these equations with $y^1 = y^1(t)$ gives

$$(3:4) \quad x_a^1 = X_a^1 + X_{y^j}^1 y_a^j.$$

Under the transformation (3:3) let

$$(3:5) \quad f(t, x, x_t) = g(t, y, y_t), \quad D(t, x, x_t) = D^*(t, y, y_t)$$

where x_t and y_t are connected by the relations (3:4). Equations (1:1) and (1:4) become respectively

$$(3:6) \quad I = \int_T g(t, y, y_t) dt$$

$$(3:7) \quad I^* = \int_T D^*(t, y, y_t) dt.$$

The family of extremals (3:1) is now replaced by equations (3:2) and the slope functions of F become

$$(3:8) \quad q_a^1(t, y) \equiv 0.$$

The Euler equations are

$$dg_{y_a}^1/dt_a = g_{y^1}^1,$$

which can be written

$$g_{y_a^1 t_a}^1 + y_a^j g_{y_a y^j}^1 + y_{ba}^j g_{y_a y_b^j}^1 = g_{y^1}^1.$$

When the family of extremals (3:2) is substituted in these equations one obtains the relations

$$(3:9) \quad g_{y_a^1 t_a}^1(t, y, 0) = g_{y^1}^1(t, y, 0).$$

In order that the region F be a field, we must have an invariant integral of the form (3:7) with the integrand satisfying the equations

$$(3:10) \quad D^*(t, y, 0) = g(t, y, 0), \quad D_{y_a}^*(t, y, 0) = g_{y_a}^1(t, y, 0).$$

From Theorem 2:1 we see that the integrand of this invariant integral must be of the form

$$D^*(t, y, y_t) = C(t, y) + \sum_{r=1}^s \frac{1}{r!} y_{a_1}^{i_1} \dots y_{a_r}^{i_r} C_{i_1 \dots i_r}^{a_1 \dots a_r}(t, y)$$

with the C 's satisfying the equations

$$(3:11) \quad (\partial/\partial t_a) C_{i_1 \dots i_r}^{a_1 \dots a_r} = C_{i_1 \dots i_r}^{a_1 \dots a_r} \quad (r = 0, 1, \dots, s),$$

where analogues of the identities (2:3) and all properties of the B 's hold for the C 's. Using equations (3:10) we have

$$(3:12) \quad C(t, y) = D^*(t, y, 0) = g(t, y, 0), \quad C_1^a(t, y) = D_{y_a}^*(t, y, 0) = g_{y_a}^1(t, y, 0).$$

The integrand function D^* becomes

(3:13) $D^*(t, y, y_t) = g(t, y, 0) + y_a^1 g_{y_a^1}(t, y, 0) + \sum_{r=2}^s \frac{1}{r!} y_{a_1}^{1_1} \dots y_{a_r}^{1_r} C_{1_1 \dots 1_r}^{a_1 \dots a_r}(t, y)$
and the conditions given by (3:11) are

$$(3:14) \quad g_{y_a^1 t_a}(t, y, 0) = g_{y_a^1}(t, y, 0),$$

$$(3:15) \quad (\partial/\partial t_a) C_{1_1 \dots 1_r}^{a_1 \dots a_r} = C_{1_1 \dots 1_r}^{a_1 \dots a_r} \quad (r = 1, \dots, s).$$

Equations (3:14) are identical with equations (3:9) and by hypothesis are satisfied.

We shall now prove two theorems the first of which is the following:

THEOREM 3:1. If $m > 1$, every region F of tx -space which is simply covered by an n -parameter family of extremals, in the sense described above, has associated with it an invariant integral I^* with which it forms a field for the integral I specified in (1:1). If the coordinates are transformed as described above the extremals of the field will be given by equations (3:2) and the slope functions by the equations (3:8). There will be an invariant integral I^* of the field in which the integrand function D_1^* is

$$(3:16) \quad D_1^*(t, y, y_t) = g(t, y, 0) + y_a^1 g_{y_a^1}(t, y, 0) + \frac{1}{2!} L_{1j}^{ab} y_{ay}^{1j},$$

where

$$(3:17) \quad m L_{1j}^{ab} = \int_{F_a}^{t_a} [K_{j1}^b + \frac{1}{m-1} \bar{K}_{j1}^b] dt_a + \int_{F_b}^{t_b} [K_{1j}^a + \frac{1}{m-1} \bar{K}_{1j}^a] dt_b,$$

$$K_{1j}^a = g_{y_a^1 y_j^1}(t, y, 0) - g_{y_j^1 y_a^1}(t, y, 0),$$

and $\bar{K}_{1j}^a$ is obtained from K_{1j}^a by replacing t_a by F_a . Furthermore, the integrand D^* of every invariant integral I^* associated with F must be of the form (3:13) with

$$(3:18) \quad C_{1_1 \dots 1_r}^{a_1 \dots a_r} = K_{1_1 \dots 1_r}^{a_1 \dots a_r} + L_{1_1 \dots 1_r}^{a_1 \dots a_r} \quad (r = 2, \dots, s).$$

Here the functions $K_{1_1 \dots 1_r}^{a_1 \dots a_r}$ are solutions of the equations

$$(3:19) \quad \frac{\partial}{\partial t_a} K_{i_1 i_2 \dots i_r}^{a a_1 \dots a_r} = 0 \quad (r = 2, \dots, s),$$

$$(3:20) \quad K_{i_1 \dots i_{s-1}}^{a_1 \dots a_s} = 0$$

with the property that the interchange of two a's or two i's simply changes the sign. Here

$$(3:21) \quad K_{i_1 \dots i_{r-1}}^{a_1 \dots a_r} = \frac{\partial}{\partial y^1} K_{i_1 \dots i_r}^{a_1 \dots a_r} - \frac{\partial}{\partial y^{i_1}} K_{i_1 i_2 \dots i_r}^{a_1 a_2 \dots a_r} - \dots \\ - \frac{\partial}{\partial y^{i_r}} K_{i_1 \dots i_{r-1} i_r}^{a_1 \dots a_{r-1} a_r}.$$

The function L_{ij}^{ab} is defined by (3:17) and when $r \geq 2$ the function $L_{i_1 \dots i_r}^{a a_1 \dots a_r}$ is defined by the formula

$$(3:22) \quad m L_{i_1 \dots i_r}^{a a_1 \dots a_r} = \int_{\bar{t}_a}^{t_a} L_{i_1 \dots i_r i}^{a_1 \dots a_r} dt_a - \int_{\bar{t}_{a_1}}^{t_{a_1}} L_{i_1 i_2 \dots i_r}^{a a_2 \dots a_r} dt_{a_1} \\ - \dots - \int_{\bar{t}_{a_r}}^{t_{a_r}} L_{i_1 \dots i_{r-1} i_r}^{a_1 \dots a_{r-1} a_r} dt_{a_r},$$

where

$$(3:23) \quad L_{i_1 \dots i_{r-1}}^{a_1 \dots a_r} = K_{i_1 \dots i_{r-1}}^{a_1 \dots a_r} + \sum \frac{k!}{\tau^k (m-1) \dots (m-k)} K_{i_1 \dots i_{r-1}}^{a_1 \dots a_r}(\tau^k),$$

the function $K_{i_1 \dots i_{r-1}}^{a_1 \dots a_r}(\tau^k)$ is obtained from $K_{i_1 \dots i_{r-1}}^{a_1 \dots a_r}$ by replacing k of the values $t_{a_1}, \dots, t_{a_r}$ by the corresponding values $\bar{t}_{a_1}, \dots, \bar{t}_{a_r}$ and the sum is taken over all possible replacements τ^k of this type in which $k \leq r$.

Here and elsewhere the a's used in the limits of integration are not to be summed.

In order to prove the first part of the theorem we must show that the conditions expressed by equations (3:15) are satisfied by the coefficients $C_{ij}^{ab} = L_{ij}^{ab}$, $C_{ij}^a = K_{ij}^a$, $C_{i_1 \dots i_r}^{a_1 \dots a_r} = 0$ in equations (3:16). These conditions take the special form

$$(3:24) \quad \frac{\partial}{\partial t_a} L_{ij}^{ab} = K_{ji}^b,$$

$$(3:25) \quad \frac{\partial L_{ij}^{ab}}{\partial y^k} - \frac{\partial L_{kj}^{ab}}{\partial y^i} - \frac{\partial L_{ik}^{ab}}{\partial y^j} = 0.$$

The equation (3:25) holds with L_{ij}^{ab} replaced by K_{ij}^a , as is seen by the use of the definition (3:17) of K_{ij}^a . In view of this fact it follows from the definition (3:17) of L_{ij}^{ab} that equation (3:25) must also hold. To establish equation (3:24) we need the equations

$$(3:26) \quad \frac{\partial}{\partial t_a} K_{ij}^a = 0, \quad K_{ij}^a = -K_{ji}^a,$$

which are easy consequences of equations (3:14) and (3:17). From equations (3:17) it is found that (a not summed)

$$\begin{aligned} m \frac{\partial L_{ij}^{ab}}{\partial t_a} &= K_{ji}^b + \frac{1}{m-1} \bar{K}_{ji}^b + \int_{t_b}^{t_b} \frac{K_{ij}^a}{t_a} dt_b & (a \neq b), \\ m \frac{\partial L_{ij}^{ab}}{\partial t_a} &= 0 & (a = b). \end{aligned}$$

Summing for a one obtains, by equations (3:26), the relation

$$m \frac{\partial L_{ij}^{ab}}{\partial t_a} = (m-1)K_{ji}^b + \bar{K}_{ji}^b - [K_{ij}^b - \bar{K}_{ij}^b] = mK_{ji}^b,$$

as was to be proved.

In order to establish the second part of Theorem 3:1 we observe that the equations

$$(3:27) \quad \frac{\partial}{\partial t_a} K_{i_1 i_2 \dots i_{r-1} 1}^{a a_2 \dots a_r} = 0 \quad (r \geq 2),$$

$$(3:28) \quad \frac{\partial}{\partial y^k} K_{i_1 \dots i_{r-1} 1}^{a_1 \dots a_r} - \frac{\partial}{\partial y^{i_1}} K_{k i_2 \dots i_{r-1} 1}^{a_1 a_2 \dots a_r} - \dots - \frac{\partial K_{i_1 \dots i_{r-1} k}^{a_1 \dots a_r}}{\partial y^1} = 0$$

hold, by virtue of equations (3:19) and (3:21). Using equations

(3:28) and (3:23) it is found that equation (3:28) holds when

$K_{i_1 \dots i_{r-1} 1}^{a_1 \dots a_r}$ is replaced by $L_{i_1 i_2 \dots i_{r-1} 1}^{a a_1 \dots a_r}$. From this fact it follows that

$$C_{i_1 \dots i_{r-1} 1}^{a_1 \dots a_r} = K_{i_1 \dots i_{r-1} 1}^{a_1 \dots a_r} \quad (r \geq 2)$$

when equation (3:18) holds, the right member being obtained by equation (3:21) by replacing K by C . By the use of this result and equations (3:18) and (3:19), the equations (3:15) reduce to

$$(3:29) \quad \frac{\partial}{\partial t_a} L_{i_1 i_2 \dots i_r}^{a a_1 \dots a_r} = K_{i_1 \dots i_r 1}^{a_1 \dots a_r} \quad (r = 2, \dots, s).$$

When the integers $a_1, \dots, a_r$ are not distinct, the right and left members of this equation are automatically zero, in view of the symmetry properties of the K 's and the L 's relative to their upper indices. When $r = s$ the first member of (3:29) is zero since the upper or the lower indices cannot all be distinct in this case. The equation (3:29) then follows from equation (3:20). It remains to discuss the case when $r < s$ and the integers $a_1, \dots, a_r$ are distinct. Since $L_{i_1 i_2 \dots i_r}^{a a_1 \dots a_r} = 0$ when a is one of the values $a_1, \dots, a_r$, the sum in (3:29) can be taken over the $m - r$ integers from 1 to m with $a_1, \dots, a_r$ deleted. Summing over these integers it is found, by formula (3:22), that

$$(3:30) \quad \begin{aligned} m \frac{\partial}{\partial t_a} L_{i_1 i_2 \dots i_r}^{a a_1 \dots a_r} &= (m - r) L_{i_1 \dots i_r 1}^{a_1 \dots a_r} - \int_{\bar{t}_{a_1}}^{t_{a_1}} \frac{\partial}{\partial t_a} L_{i_1 i_2 \dots i_r 1}^{a a_2 \dots a_r} dt_{a_1} \\ &\quad - \dots - \int_{\bar{t}_{a_r}}^{t_{a_r}} L_{i_1 \dots i_r 1}^{a_1 \dots a_{r-1} a} dt_{a_r}. \end{aligned}$$

In order to compute the value of the first integral in (3:30) we observe that in equation (3:27) the sum may be taken over the $m - r + 1$ values distinct from $a_2, \dots, a_r$. Using this fact together with equations (3:23) and (3:27) and summing for a distinct from $a_1, \dots, a_r$ we find by integration that

$$(3:31) \quad \int_{\bar{t}_{a_1}}^{t_{a_1}} \frac{\partial L_{i_1 i_2 \dots i_r 1}^{a a_2 \dots a_r}}{\partial t_a} dt_{a_1} = - \left[K_{i_1 \dots i_r 1}^{a_1 \dots a_r} + \sum_{\mathcal{Z}_1^k} \frac{k!}{(m-1) \dots (m-k)} K_{i_1 \dots i_r 1}^{a_1 \dots a_r} (\mathcal{Z}_1^k) \right]_{\bar{t}_{a_1}}^{t_{a_1}},$$

where $\mathcal{Z}_1^k$ denotes that k of the values $t_{a_2}, \dots, t_{a_r}$ have been replaced by the corresponding values $\bar{t}_{a_2}, \dots, \bar{t}_{a_r}$, and the sum is taken for all possible replacements $\mathcal{Z}_1^k$ of this type. By the use of formula (3:31) and similar ones for the remaining integrals in (3:30), it is found that the equation (3:30) reduces to (3:29), except for the factor m . The equation (3:29) therefore holds and

the theorem is proved.

Corresponding to Theorem 3:1, when $m = 1$, we have the following theorem.

THEOREM 3:2. If $m = 1$, every region F of the tx -space which is simply covered by an n -parameter family of extremals, in the sense described above, has associated with it an invariant integral I^* with which it forms a field for the integral I specified in (1:1) if and only if, when the coordinates are transformed as described above, the equations

$$(3:32) \quad g_{\frac{1}{y}y^j}(t,y,0) = g_{\frac{1}{y^jy}}(t,y,0) \quad (i,j=1,\dots,n)$$

are satisfied. Then, the extremals of the field will be given by (3:2), the slope functions by the equations (3:8) and there will be an invariant integral I^* of the field in which the integrand function D^* is

$$(3:33) \quad D^*(t,y,y_t) = g(t,y,0) + y_1^1 g_{y_1^1}(t,y,0).$$

If $m = 1$, equations (3:15) with $r = 2, \dots, s$ do not appear and, by means of equations (3:12), equations (3:15) with $r = 1$ reduce to equations (3:32), hence the theorem. In this theorem it is evident that equations (3:32) are always satisfied for $n = 1$.

4. Results expressed in terms of the original coordinates.

In the last section we obtained results in terms of specialized coordinates (t,y) , which are related to the coordinates (t,x) by the transformation (3:3). We shall use the inverse transformation

$$(4:1) \quad y^1 = Y^1(t,x)$$

in expressing these results in terms of the original coordinates (t,x) .

We shall now derive several sets of identities. Substituting equations (3:3) into relations (4:1), and vice versa, we find

$$(4:2) \quad y^i = Y^i[t, X(t, y)], \quad x^i = X^i[t, Y(t, x)].$$

Differentiating these identities with respect to t_a we have

$$(4:3) \quad 0 = Y_a^i + Y_{x^j}^i p_a^j, \quad 0 = X_a^i + X_{y^j}^i Y_a^j.$$

Differentiating the first group of identities (4:2) with respect to y^j and the second group with respect to x^j we obtain

$$(4:4) \quad S_j^i = Y_{x^k}^i X_{y^j}^k, \quad S_j^i = X_{y^k}^i Y_{x^j}^k,$$

where $S_j^i = 1$ when $i = j$ and $S_j^i = 0$ for $i \neq j$.

Differentiation with respect to t_a of equation (4:1) with $x^j = x^j(t)$ gives

$$y_a^i = Y_a^i + Y_{x^j}^i x_a^j.$$

Using the first group of identities (4:3), these equations become

$$(4:5) \quad y_a^i = (x_a^j - p_a^j) Y_{x^j}^i.$$

By means of equations (3:3), (3:4) and (3:5) we find

$$(4:6) \quad g(t, y, y_t) = f(t, X, X_t + X_{y_t} y_t).$$

Making use of equations (3:1), (3:2) and (4:6), we obtain the relations

$$(4:7) \quad g(t, y, 0) = f(t, x, p), \quad g_{y_a^i}(t, y, 0) = X_{y^i}^j f_{x_a^j}(t, x, p).$$

From relations (4:4), (4:5) and (4:7) we find

$$(4:8) \quad y_a^i g_{y_a^i}(t, y, 0) = (x_a^i - p_a^i) f_{x_a^i}(t, x, p).$$

Under the transformation (4:1) let us set

$$(4:9) \quad C_{i_1 \dots i_r}^{a_1 \dots a_r}(t, Y) = X_{y i_1}^{j_1} \dots X_{y i_r}^{j_r} A_{j_1 \dots j_r}^{a_1 \dots a_r}(t, x) \quad (r = 2, \dots, s).$$

By means of equations (3:5), (4:4), (4:7), (4:8) and (4:9), we now have the following lemma.

LEMMA 4:1. The integrand function D of an invariant integral I^* of the form (1:4) must have the form

$$(4:10) \quad \begin{aligned} D(t, x, x_t) = & f(t, x, p) + (x_a^1 - p_a^1) f_{x_a^1}(t, x, p) \\ & + \sum_{r=2}^s \frac{1}{r!} (x_{a_1}^{i_1} - p_{a_1}^{i_1}) \dots (x_{a_r}^{i_r} - p_{a_r}^{i_r}) A_{i_1 \dots i_r}^{a_1 \dots a_r}(t, x), \end{aligned}$$

where

$$A_{i_1 \dots i_r}^{a_1 \dots a_r}(t, x) = Y_{x i_1}^{j_1} \dots Y_{x i_r}^{j_r} C_{j_1 \dots j_r}^{a_1 \dots a_r}(t, Y).$$

On transforming back to the original coordinates (t, x) , Theorem 3:1 becomes

THEOREM 4:1. If $m > 1$, every region F of tx-space which is simply covered by an n-parameter family of extremals (3:3), in the sense described at the beginning of Section 3, has associated with it an invariant integral I^* with which it forms a field for the integral I specified in (1:1). The slope functions of the field will be given by the equations

$$(4:11) \quad p_a^1(t, x) = x_a^1[t, Y(t, x)].$$

There will be an invariant integral I^* of the field in which the integrand function D is of the form (4:10) with the A's for which $r > 2$ identically zero and

$$A_{ij}^{ab} = Y_{x^1}^k Y_{x^j}^h L_{kh}^{ab}$$

where L_{ij}^{ab} is given by formula (3:17) with

$$K_{ij}^a = \frac{\partial}{\partial y^j} [x_{y^1}^k f_{x_a^k}(t, x, p)] - \frac{\partial}{\partial y^1} f[x_{y^j}^h f_{x_a^h}(t, x, p)].$$

Furthermore the integrand of every invariant integral I^* associated with F must be of the form (4:10) with

$$A_{ij}^{ab} = Y_{xi}^k Y_{xj}^h [K_{kh}^{ab} + L_{kh}^{ab}]$$

and for $r = 3, \dots, s$

$$A_{i_1 \dots i_r}^{a_1 \dots a_r} = [Y_{x i_1}^{j_1} \dots Y_{x i_r}^{j_r}] [K_{j_1 \dots j_r}^{a_1 \dots a_r} + L_{j_1 \dots j_r}^{a_1 \dots a_r}],$$

where the K 's and the L 's are the functions defined in Theorem 3:1.

Here, as elsewhere, i, j, k and h , with or without subscripts, range over the integers 1 to n . Similarly a and b range from 1 to m . Any of these letters used in indicating limits of integration are not to be summed.

By means of relations (4:7), the equations (3:32) become

$$X_{y1}^k (\partial / \partial y^j) f_{x1}^k(t, x, p) = X_{y1}^h (\partial / \partial y^j) f_{x1}^h(t, x, p).$$

These relations may be written

$$X_{y1}^k X_{y1}^{k_1} (\partial / \partial x^{k_1}) f_{x1}^h(t, x, p) = X_{y1}^h X_{y1}^{h_1} (\partial / \partial x^{h_1}) f_{x1}^h(t, x, p).$$

On multiplying by $Y_{x1}^{i_1} Y_{x1}^{j_1}$ and using the relations (4:4) we have

$$(4:12) \quad (\partial / \partial x^j) f_{x1}^j(t, x, p) = (\partial / \partial x^i) f_{x1}^i(t, x, p).$$

Theorem 3:2 can now be stated in the following manner.

THEOREM 4:2. If $m = 1$ every region F of tx -space which is simply covered by an n -parameter family of extremals (3:3), in the sense described at the beginning of Section 3, has associated with

it an invariant integral I^* with which it forms a field for the integral I specified in (1:1) if and only if the equations (4:12) are satisfied. Then, the slope functions of the field will be given by the equations (4:11) and there will be an invariant integral I^* of the field in which the integrand function D is of the form (4:10) where all of the A 's are identically zero.

5. Certain systems of partial differential equations. We shall first consider a determinant of the form

$$(5:1) \quad \begin{vmatrix} \partial / \partial t_a \\ \partial U^q / \partial t_a \end{vmatrix} \quad (a = 1, \dots, m; q = 2, \dots, m),$$

where the U 's are arbitrarily selected functions having continuous partial derivatives of the second order with respect to t . The elements $\partial / \partial t_a$ of the first row are differential operators and they operate on the elements of their cofactors according to the usual rules for differentiating a determinant or a product. We shall prove the following lemma.

LEMMA 5:1. Any determinant of the form (5:1) has the value zero.

To prove this lemma, let us expand the determinant in terms of elements of the first row. The determinant (5:1) is then equal to

$$\sum_{a=1}^m (-1)^{a-1} \left(\partial / \partial t_a \right) \begin{vmatrix} \partial U^q / \partial t_1 \dots \partial U^q / \partial t_{a-1} & \partial U^q / \partial t_{a+1} \dots \partial U^q / \partial t_m \end{vmatrix}.$$

By operating with the element $\partial / \partial t_a$ on the columns of its minor, in accordance with the usual rules for differentiating a determinant, then moving the column on which we have operated over the ones which precede it, the value of the determinant (5:1) is a sum of terms of the form.

$$M = (-1)^{a-1+b-1} \left[\frac{\partial^2 U^q}{\partial t_b \partial t_a} \quad \frac{\partial U^q}{\partial t_e} \right] \quad \text{for } b < a,$$

plus a sum of terms of the form

$$N = (-1)^{a-1+b-2} \left[\frac{\partial^2 U^q}{\partial t_b \partial t_a} \quad \frac{\partial U^q}{\partial t_e} \right] \quad \text{for } b > a,$$

with e ranging over the integers $1, \dots, m$ excluding a and b . For every possible pair of values of a and b giving a term M there is a corresponding term N obtained by interchanging the values of a and b , and conversely. Moreover, these corresponding values of M and N are equal but opposite in sign. Hence the value of the determinant (5:1) is zero and the lemma is proved.

We shall now prove the following theorem.

THEOREM 5:1. The equations

$$(5:2) \quad \left(\frac{\partial}{\partial t_a} \right) K^{a a_2 \dots a_r} = 0 \quad (a_2 < \dots < a_r; r = 2, \dots, s < m),$$

in which the interchange of two a 's simply changes the sign of the K 's, are satisfied if we set

$$(5:3) \quad K^{a_1 \dots a_r} \quad (a_1 < \dots < a_r)$$

equal to $(-1)^{a_1 + \dots + a_r}$ multiplied by the determinant of the matrix which remains when the rows $a_1, \dots, a_r$ are deleted from the $(m-r)$ -rowed matrix

$$\left\| \frac{\partial U^v}{\partial t_a} \right\| \quad (a = 1, \dots, m; v = r+1, \dots, m),$$

where the U 's are arbitrarily selected functions having continuous partial derivatives of the second order with respect to t .

Let us consider the determinant

$$(5:4) \quad \begin{vmatrix} \partial / \partial t_a \\ \vdots \\ V_a \\ \partial U^v / \partial t_a \end{vmatrix} \quad (a=1, \dots, m; \quad d=2, \dots, r; \quad v=r+1, \dots, m),$$

where the V 's are arbitrarily selected functions independent of t . This determinant has the value zero, since it is of the form (5:1) with $U^d = V_a^d t_a$. We shall denote by $V_{a_2 \dots a_r}$ the determinant formed by selecting the columns $a_2, \dots, a_r$ from the matrix of V 's where $a_2 < \dots < a_r$. When the elements of the first row operate on the V 's the result is zero since the V 's are independent of t .

We shall now expand the determinant (5:4) in terms of minors of the first r rows and then expand these r -rowed minors in terms of their first rows. Using the functions (5:3) the value of the above determinant, except perhaps for sign, becomes

$$\begin{aligned} & [V_{a_2 \dots a_r} (\partial / \partial t_{a_1}) + \dots + (-1)^{r-1} V_{a_1 \dots a_{r-1}} (\partial / \partial t_{a_r})] K^{a_1 \dots a_r} \\ &= V_{a_2 \dots a_r} (\partial / \partial t_{a_1}) K^{a_1 \dots a_r} + \dots + V_{a_1 \dots a_{r-1}} (\partial / \partial t_{a_r}) K^{a_r a_1 \dots a_{r-1}} \\ &= V_{a_2 \dots a_r} (\partial / \partial t_a) K^{a a_2 \dots a_r}, \end{aligned}$$

where in these summations a without a subscript is not restricted but the a 's with subscripts range only over sets of values satisfying the conditions $a_1 < \dots < a_r$. Thus we see that the coefficients of the determinants $V_{a_2 \dots a_r}$, whose elements are arbitrarily selected functions independent of t , are the left sides of equations (5:2) with the K 's having the values given by the functions (5:3). Since the value of the determinant (5:4) is zero, as observed above, and since the V 's are arbitrary, it follows that equations (5:2) are satisfied by the functions (5:3) and the theorem is proved.

We shall now determine K's which will satisfy the conditions of Theorem 3:1. Since the interchange of two a's or two i's simply changes the sign all of the K's with indices $a_1, \dots, a_r$ and $i_1, \dots, i_r$ are completely determined by those for which $a_1 < \dots < a_r$ and $i_1 < \dots < i_r$, all of the equations (3:19) are satisfied when those for which $a_2 < \dots < a_r$ and $i_1 < \dots < i_r$ are satisfied and all of the equations (3:20) are satisfied when those for which $a_1 < \dots < a_s$ and $i < i_1 \dots < i_s$ are satisfied.

It is convenient to consider first K's which satisfy equations (3:19) when $s < m$. In this case $s = n$ and it can readily be shown from the properties of the K's that equations (3:20) are always satisfied. The equations (3:19) when $s < m$ will be satisfied by K's chosen according to the following theorem.

THEOREM 5:2. The equations

$$(5:5) \quad (\partial / \partial t_a) K_{1_1 i_2 \dots i_r}^{a_1 a_2 \dots a_r} = 0 \quad (a_2 < \dots < a_r; \quad i_1 < \dots < i_r; \quad r = 2, \dots, s < m),$$

in which the interchange of two a's or two i's simply changes the sign of the K's, are satisfied if we set

$$(5:6) \quad K_{1_1 \dots i_r}^{a_1 \dots a_r} \quad (a_1 < \dots < a_r; \quad i_1 < \dots < i_r)$$

equal to $(-1)^{a_1 + \dots + a_r}$ multiplied by the determinant of the matrix which remains when the rows $a_1, \dots, a_r$ are deleted from the $(m-r)$ -rowed matrix

$$\left| \left(\partial / \partial t_a \right) U_{1_1 \dots i_r}^v \right| \quad (v = r + 1, \dots, m)$$

where the U's are arbitrarily selected functions having continuous partial derivatives of the second order with respect to t and as many other continuous partial derivatives as are needed for the calculus of variations problem under consideration.

For a fixed set of values of $i_1, \dots, i_r$ the equations (5:5) are of the form (5:2) and the functions (5:6) are of the form (5:3). Hence, this theorem follows at once from Theorem 5:1.

In determining K 's which will satisfy the conditions of Theorem 3:1 with $s = m = n$, as before, we observe that equations (3:20) are always satisfied. Equations (3:19) with $r = 2, \dots, s-1$ will be satisfied by K 's chosen according to Theorem 5:2. We can select K 's that will satisfy equations (3:19) with $r = s = m = n$ by means of the following theorem.

THEOREM 5:3. When $s = m = n$, the equations

$$(5:7) \quad (\partial / \partial t_a) K_{1_1 i_2 \dots i_s}^{a a_2 \dots a_m} = 0 \quad (a_2 < \dots < a_m; i_1 < \dots < i_s),$$

in which the interchange of two a 's or two i 's simply changes the signs of the K 's, are satisfied if we set

$$(5:8) \quad K_{1_1 \dots i_s}^{1 \dots m} = V_{1_1 \dots i_s} \quad (i_1 < \dots < i_s),$$

where the V 's are arbitrarily selected functions independent of t having continuous partial derivatives of as many orders as are needed for the calculus of variations problem under consideration.

This theorem follows at once since the left sides of equations (5:7) all vanish when the K 's from equation (5:8) are substituted.

We shall now consider K 's which will satisfy the conditions of Theorem 3:1 when $s = m < n$. Equations (3:19) with $r = 2, \dots, s-1$ will be satisfied by K 's chosen according to Theorem 5:2. Equations (3:19) with $r = s = m$ will be satisfied by K 's determined by means of Theorem 5:3 but we must make these same K 's satisfy equations (3:20). This can be done by the use of the following theorem.

THEOREM 5:4. When $s = m < n$, the equations

$$(5:9) \quad \begin{aligned} & (\partial/\partial y^1)v_{i_1 \dots i_s} - (\partial/\partial y^{i_1})v_{1i_2 \dots i_s} \\ & - \dots - (\partial/\partial y^{i_s})v_{i_1 \dots i_{s-1}1} = 0 \quad (1 < i_1 < \dots < i_s), \end{aligned}$$

where the interchange of two i 's simply changes the sign of the V 's, are satisfied if we set

$$(5:10) \quad v_{i_1 \dots i_s} \quad (i_1 < \dots < i_s)$$

equal to the determinant of the matrix obtained by selecting the rows $i_1, \dots, i_s$ from the s -rowed matrix

$$\left\| \left(\partial/\partial y^j \right) W_{i_1 \dots i_s}^u \right\| \quad (j = 1, \dots, n; u = 1, \dots, s),$$

where the W 's are arbitrarily selected functions independent of t having continuous partial derivatives of the second order with respect to y and as many other continuous partial derivatives as are needed for the calculus of variations problem under consideration.

To prove this theorem let us consider a determinant of the form

$$(5:11) \quad \begin{vmatrix} \partial/\partial y^f \\ (\partial/\partial y^f)W_{i_1 \dots i_s}^u \end{vmatrix} \quad (f = 1, i_1, \dots, i_s; u = 1, \dots, s).$$

This determinant is of the form (5:1), with W 's and y 's in place of U 's and t 's respectively, and hence it has the value zero.

Let us expand the determinant (5:11) in terms of elements of the first row. Making use of the functions (5:10), the determinant becomes

$$(\partial/\partial y^1)v_{1_1\dots 1_s} - (\partial/\partial y^{1_1})v_{11_2\dots 1_s} + \dots + (-1)^s(\partial/\partial y^s)v_{11_1\dots 1_{s-1}}$$

We see that these expressions and the left sides of equations (5:9) are equal, since the interchange of two 1's simply changes the sign of the V's. As observed above, the determinant (5:11) has the value zero and hence the functions (5:10) satisfy equations (5:9), thus proving the theorem.

The conditions which the K's in Theorem 4:1 must satisfy are the same as those which the K's in Theorem 3:1 must satisfy. We have shown how these conditions can always be satisfied by choosing the K's in accordance with Theorem 5:2, Theorem 5:3 and Theorem 5:4.

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VITA

Aubrey Wilfred Landers, Jr., was born in Manchester, New Hampshire, on September 2, 1906. After attending school for one year in Winchester, Massachusetts, he received the remainder of his pre-college education in Middleton, Nova Scotia. He received the degree of Bachelor of Arts from Acadia University, Wolfville, Nova Scotia, in May, 1926. For the next three years, he did part-time teaching and graduate work at Brown University, where he received the degree of Master of Science in June, 1929. During the college year 1929-30, he was an instructor in mathematics at Hunter College of the City of New York. In 1930, he was transferred to the newly formed Brooklyn College of the City of New York, where he is still located. He has been in residence at the University of Chicago for five consecutive quarters beginning with the Summer Quarter of 1933 and for the Summer Quarters of 1935, 1937, 1938, and 1939. At Acadia University, he studied under Professor Jeffery; at Brown University, under Professors Adams, Archibald, Bennett, Ingraham, Langer, Richardson and Tamarkin; at the University of Chicago, under Professors Bartky, Bliss, Dickson, Graves, Hestenes, Lane, Logsdon, McShane, and Reid. To all of his instructors he wishes to express his appreciation. To Professors Bliss and Hestenes, who directed this thesis, he is particularly indebted and wishes to express his sincere gratitude.

